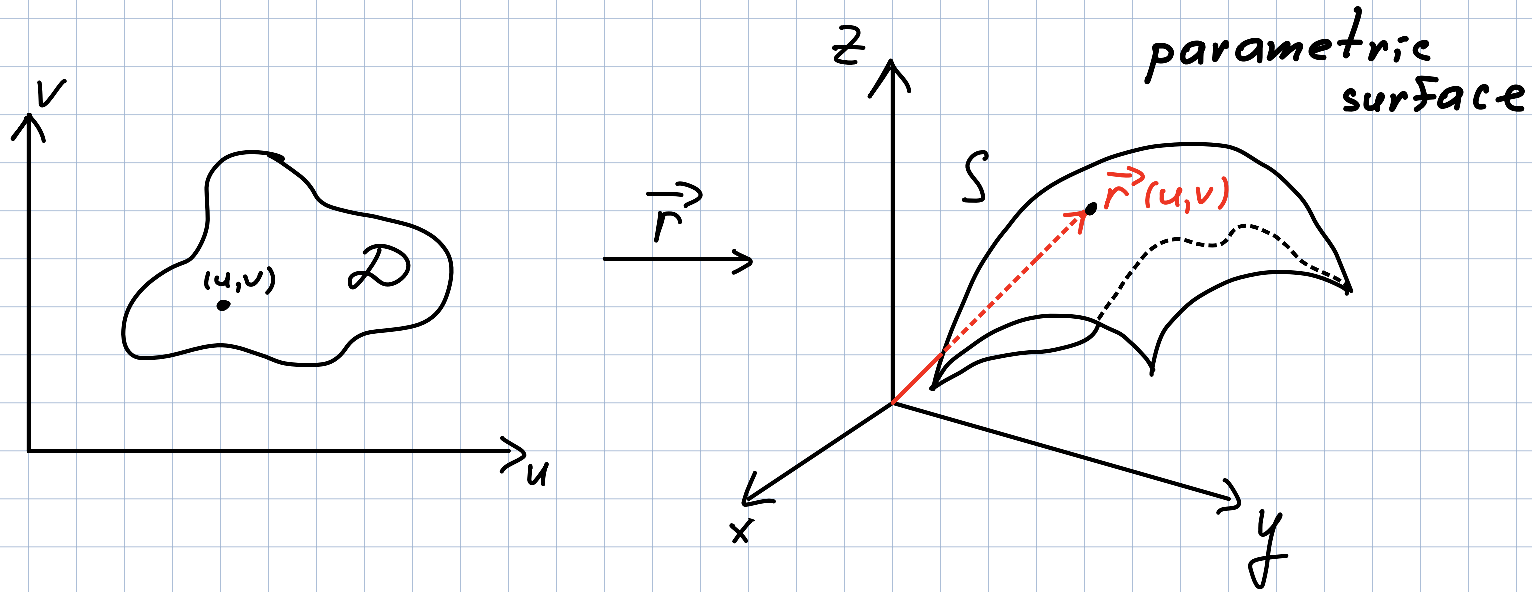
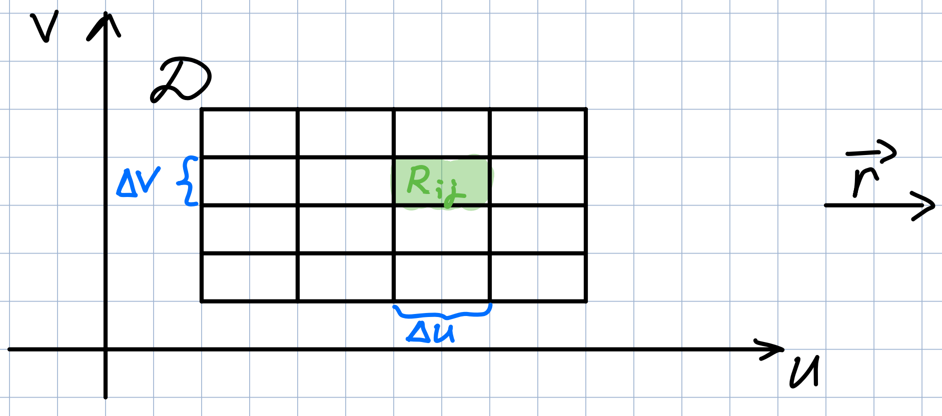
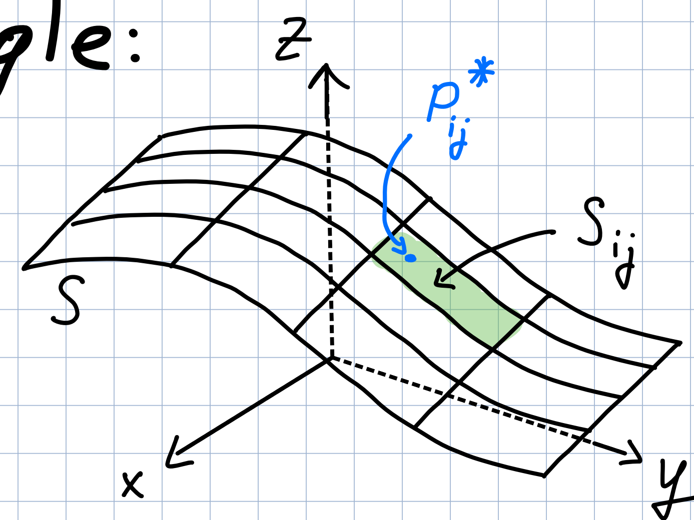
Surface integrals

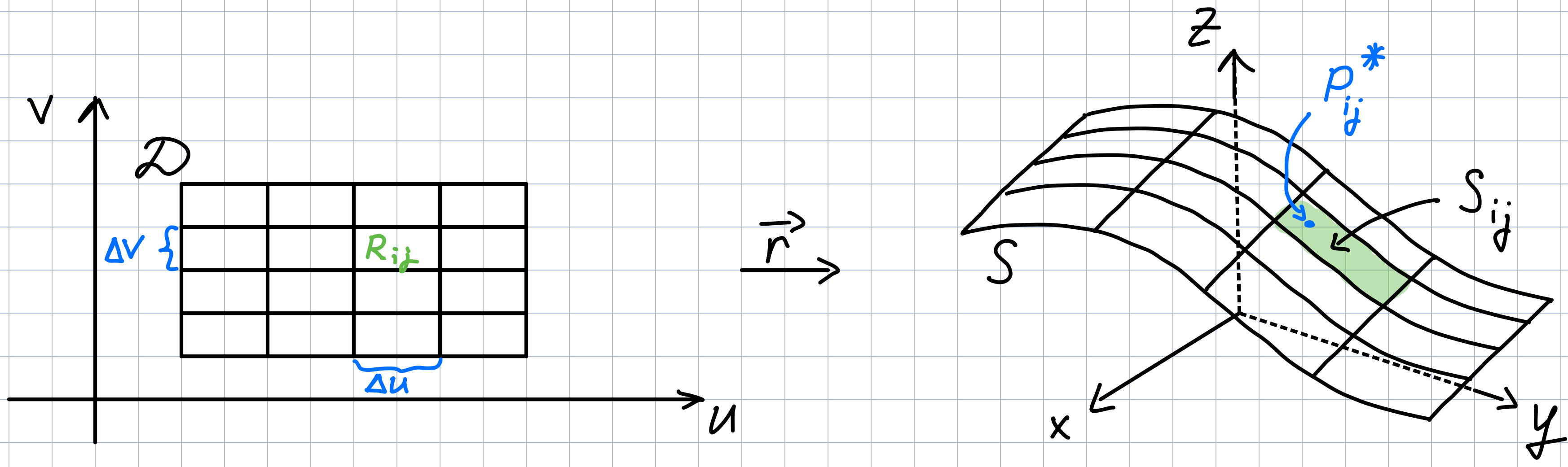


$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k} \qquad (u,v) \in \mathcal{D}$$

Assume $\mathcal{D}$ is a rectangle:







Consider the Riemann sum:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}$$

The surface integral of f over the surface S:

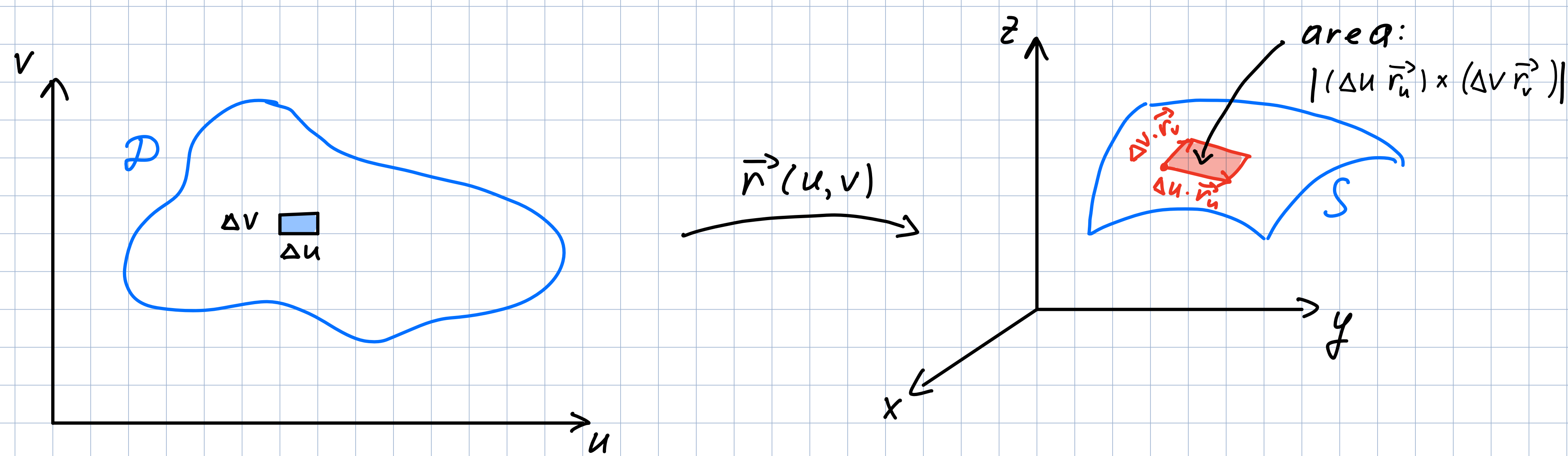
$$\iint_S f(x,y,z)\,dS = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*) \Delta S_{ij}$$

Recall:

$$\Delta S_{ij} \approx |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v,$$

where
$$\vec{r}_u = \frac{\partial x}{\partial u}\vec{i} + \frac{\partial y}{\partial u}\vec{j} + \frac{\partial z}{\partial u}\vec{k},$$
$$\vec{r}_v = \frac{\partial x}{\partial v}\vec{i} + \frac{\partial y}{\partial v}\vec{j} + \frac{\partial z}{\partial v}\vec{k}.$$

tangent vectors at a corner of S_{ij}



Therefore,

$$\iint_S f(x,y,z)\,dS = \iint_{\mathcal{D}} f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

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Rmk:

$$\iint_S 1\,dS = \iint_{\mathcal{D}} |\vec{r}_u \times \vec{r}_v|\,dA = \mathrm{Area}(S).$$

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Sol: 1) Parametrize the surface

$$\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = \cos\varphi \end{cases} \quad \text{with} \quad \begin{matrix} 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{matrix}$$

$$/\!/ \quad \vec{r}(\varphi,\theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$$

2) Compute $|\vec{r}_\varphi \times \vec{r}_\theta|$

$$\vec{r}_\varphi = \langle \cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi \rangle$$
$$\vec{r}_\theta = \langle -\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0 \rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \langle 0 + \sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta - 0, \sin\varphi\cos\varphi \rangle$$

$$\Rightarrow |\vec{r}_\varphi \times \vec{r}_\theta| = \sqrt{\underbrace{\underbrace{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta}_{\sin^4\varphi(\cos^2\theta+\sin^2\theta)} + \sin^2\varphi\cos^2\varphi}_{\sin^2\varphi(\sin^2\varphi+\cos^2\varphi)}} = \sin\varphi$$

3) Apply the formula

$$\iint_S x^2\,dS = \iint_D (\underbrace{\sin\varphi\cos\theta}_{x})^2 \underbrace{\sin\varphi}_{|\vec{r}_u\times\vec{r}_v|}\,dA$$

4) Evaluate the integral

$$\iint_D (\sin\varphi\cos\theta)^2\sin\varphi\,dA = \int_0^{2\pi}\int_0^{\pi}\sin^3\varphi\cos^2\theta\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\underbrace{\cos^2\theta}_{\frac{1+\cos(2\theta)}{2}}\,d\theta\int_0^{\pi}\underbrace{\sin^3\varphi}_{\sin\varphi(1-\cos^2\varphi)}\,d\varphi = \left(\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\Big|_{\theta=0}^{\theta=2\pi}\right) \times \int_0^{\pi}(\sin\varphi - \sin\varphi\cos^2\varphi)\,d\varphi$$

$$= \pi\cdot\left(-\cos\varphi + \frac{1}{3}\cos^3\varphi\Big|_{\varphi=0}^{\varphi=\pi}\right) = \frac{4\pi}{3}.$$

Rmk: If a thin sheet has the shape of a surface S and the density $\rho(x,y,z)$, then the total mass of the sheet is

$$m = \iint_S \rho(x,y,z)\,dS$$

and the center of mass is $(\bar{x},\bar{y},\bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\,\rho(x,y,z)\,dS$$

$$\bar{y} = \frac{1}{m}\iint_S y\,\rho(x,y,z)\,dS$$

$$\bar{z} = \frac{1}{m}\iint_S z\,\rho(x,y,z)\,dS$$

Graphs of Functions

S given by $z = g(x,y)$ has a natural parametrization

$$\vec{r}(x,y) = \langle x, y, g(x,y) \rangle$$

with

$$\begin{cases} \vec{r}_x = \langle 1, 0, g_x \rangle \\ \vec{r}_y = \langle 0, 1, g_y \rangle \end{cases} \Rightarrow$$

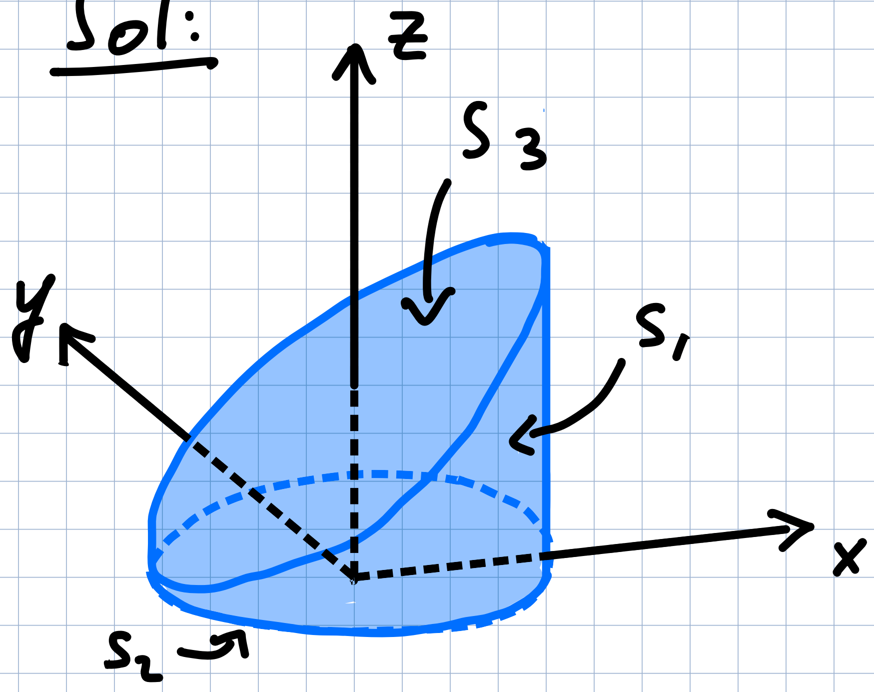
$$\vec{r}_x \times \vec{r}_y = \langle -g_x, -g_y, 1 \rangle = \langle -z_x, -z_y, 1 \rangle$$

$$\Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}.$$ Hence

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

Ex: Evaluate $\iint_S z\,dS$, where S is the surface whose sides S_1 are given by the cylinder $x^2+y^2=1$; whose bottom S_2 is the disk $x^2+y^2\le 1$ in the plane $z=0$; and whose top S_2 is the part of the plane $z=1+x$ that lies above S_2.

Sol:



S_1: $x^2+y^2=1$

S_2: $x^2+y^2\le 1$, $z=0$

S_3: $z=1+x$

1) $$\iint_S z\,dS = \iint_{S_1} z\,dS + \iint_{S_2} z\,dS + \iint_{S_3} z\,dS$$

2) Parametrize S_1: $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = z \end{cases}$ $\qquad 0 \le \theta \le 2\pi$, $\quad 0 \le z \le \underbrace{1+x}_{1+\cos\theta}$

$$\vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$$

$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle \text{ and } \vec{r}_z = \langle 0, 0, 1 \rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_z| = 1$$

$$\iint_{S_1} z\,dS = \iint_{\mathcal{D}} z \cdot 1\,dA = \int_0^{2\pi} \int_0^{1+\cos\theta} z\,dz\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta = \frac{1}{2}\int_0^{2\pi} \Big(1 + 2\cos\theta + \underbrace{\cos^2\theta}_{= \frac{1+\cos 2\theta}{2}}\Big) d\theta$$

$$= \frac{1}{2}\left(\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right)\Bigg|_{\theta=0}^{\theta=2\pi} = \frac{3\pi}{2}$$

3) S_2 lies in the plane $z=0$ $\Rightarrow$

$$\iint_{S_2} z \, dS = \iint_{S_2} 0 \, dS = 0$$

4) S_3 lies above unite disk and is part of the plane $z = 1+x$. $\Rightarrow$ $S_3 : \begin{cases} x = x \\ y = y \\ z = \underbrace{1+x}_{g(x,y) = z(x,y)} \end{cases}$ with $x^2+y^2 \le 1$

$$\iint_{S_3} z\, dS = \iint_{D} (1+x)\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

$$= \iint_{D} (1+x)\sqrt{1+0+1}\, dA = \sqrt{2}\iint_{D} (1+x)\, dA$$

$$\overset{\text{polar}}{\underset{\text{coord.}}{=}} \sqrt{2}\int_0^{2\pi}\underbrace{\int_0^1 (1+r\cos\theta)\, r\, dr}_{\frac{1}{2}r^2 + \frac{1}{3}r^3\cos\theta \Big|_{r=0}^{r=1}}\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left(\frac{1}{2} + \frac{1}{3}\cos\theta\right) d\theta = \sqrt{2}\left(\frac{\theta}{2} + \frac{\sin\theta}{3}\right)\Bigg|_{\theta=0}^{\theta=2\pi} = \sqrt{2}\,\pi$$

5) $$\iint_{S} z\, dA = \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi = \left(\frac{3}{2} + \sqrt{2}\right)\pi$$